

ON d -INVARIANTS AND GENERALISED KANENOBU KNOTS

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Abstract

We prove that for particular infinite families of L -spaces, arising as branched double covers, the d -invariants defined by Ozsváth and Szabó are arbitrarily large and small. As a consequence, we generalise a result by Greene and Watson by proving, for every odd number $\Delta \geq 5$, the existence of infinitely many non-quasi-alternating homologically thin knots with determinant Δ^2 , and a result by Hoffman and Walsh concerning the existence of hyperbolic weight 1 manifolds that are not surgery on a knot in S^3 .

INTRODUCTION

For a large family of 3-braids β , Watson constructed in [Wat06] knots $K_\beta(p, q)$ (with $p, q \in \mathbb{Z}$) with the same determinant, called generalised Kanenobu knots with braid β . He then defined subfamilies which also have the same Khovanov homology. Later, a result by Greene and Watson (cf. [GW11]) and a result by Hedden and Watson (cf. [HW14]) allowed to find infinite families of generalised Kanenobu knots which additionally share the odd-Khovanov and the knot Floer homologies. Greene and Watson used this construction in [GW11] to prove that there is a family of homologically thin knots such that the d -invariants of their double branched covers are not bounded from below. This allowed them to prove that infinitely many knots in that family are not quasi-alternating. Greene first provided an example of a non-quasi-alternating thin knot in [Gre10]: subsequently the work of Greene and Watson produced infinitely many examples of non-quasi-alternating thin knots.

In the present paper, we study other families of generalised Kanenobu knots. Our main result is the following theorem concerning the d -invariants defined by Ozsváth and Szabó in [OSz03].

Theorem 1. *For every $n \geq 2$ there exists a collection of L -spaces $\{\Sigma_m\}_{m \in \mathbb{Z}}$ satisfying $|\mathrm{H}_1(\Sigma_m; \mathbb{Z})| = (2n+1)^2$, such that the d -invariants do not admit a bound from above or below.*

A more precise statement of Theorem 1 can be found in Section 4. The L -spaces Σ_m arise as branched double covers of families of generalised Kanenobu knots. Incidentally,

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knots in the same family have the same homological invariants (Khovanov and odd-Khovanov homologies with \mathbb{Z} -coefficients, knot Floer homology with $\mathbb{Z}/2$ -coefficients) and are thin: the reduced Khovanov and odd-Khovanov homologies with \mathbb{Z} -coefficients and the knot Floer homology with $\mathbb{Z}/2$ -coefficients are free modules and are supported in a single δ -grading. Thus, as a first application of Theorem 1, we prove the following theorem.

Theorem 2. *For every odd number $\Delta \geq 5$, there exist infinite families of non-quasi-alternating thin knots with determinant Δ^2 and the same homological invariants.*

Theorem 2 generalises [GW11, Theorem 2], which proved the existence of such a family when $\Delta = 5$. The technique we use to prove Theorems 1 and 2, introduced by Greene and Watson in [GW11], relies on a relation (obtained by combining a result of Mullins in [Mul93] and a result of Rustamov in [Rus04]) between the d -invariant of a particular $\text{Spin}^{\mathbb{C}}$ structure and the Turaev torsion of the same $\text{Spin}^{\mathbb{C}}$ structure (cf. Proposition 6). Thanks to this relation (that holds when the branched double cover is an L -space), the computation of the Turaev torsion is sufficient to determine the d -invariants and prove the theorems. Another computational method to prove non-quasi-alternating-ness has recently been provided by Qazaqzeh and Chbili, and refined by Teragaito (cf. [QC14, Ter14]).

Another application of Theorem 1 concerns weight 1 manifolds that are not surgeries on a knot in S^3 . A manifold is *weight 1* if its fundamental group is the normal closure of a single element. Every manifold which is surgery on a knot is weight 1. A natural question is whether the converse holds (cf. [AFW12, Question 9.23]). A negative answer was given in [BL90] by Boyer and Lines, who exhibited an infinite set of small Seifert fibred spaces that are weight 1 but that are not surgery on a knot in S^3 . In [Doi12] Doig used the d -invariants as an obstruction for a manifold to being a surgery on a knot, and gave more examples of small Seifert fibred spaces which are weight 1 but are not surgery on a knot. After Doig, Hoffman and Walsh proved that the family of manifolds M_n from [GW11] are hyperbolic, weight 1 and are not surgery on a knot (cf. [HW13, Theorem 4.4]). As a further application of Theorem 1 we can generalise their result by proving the following theorem.

Theorem 3. *For every odd integer $\Delta \gg 0$, there exist infinitely many hyperbolic, weight 1 manifolds $M_{\Delta,p}$ with $|\mathrm{H}_1(M_{\Delta,p})| = \Delta^2$ that are not surgery on a knot in S^3 .*

The manifolds M_n studied by Hoffman and Walsh in [HW13, Theorem 4.4] satisfied $|\mathrm{H}_1(M_n)| = 25$.

As a final remark, we inform of the recent paper by Hom, Karakurt and Lidman (cf. [HKL14]), where they give further examples of Seifert fibred spaces which are weight 1 and are not surgery on a knot by using a new obstruction, coming again from the d -invariants.

Organisation. Section 1 of this paper is devoted to giving the definition and the main properties of the generalised Kanenobu knots, and to proving the relation between the

d -invariants and the Turaev torsion, expressed in Proposition 6. In Section 2 we give a presentation of the fundamental group and of the first homology group of the branched double cover of some generalised Kanenobu knots. These presentations are then used in Section 3 to compute the Turaev torsion and to prove that its coefficients are unbounded for some families of knots. In Section 4 we deduce the unboundedness of the d -invariants from the unboundedness of the coefficients of the Turaev torsion, and we prove Theorems 1 and 2. Lastly, in Section 5 we prove Theorem 3.

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1. THE GENERALISED KANENOBU KNOTS

Definition 4. Let β be a braid with 3 strands, and let β^{-1} represent the inverse of β in B_3 . Suppose that for every p and q in \mathbb{Z} the link $K_\beta(p, q)$ in Figure 1 has one component. Then we define it to be the *generalised Kanenobu knot* $K_\beta(p, q)$.

Note that the generalised Kanenobu knots are a particular case of Watson's knots (cf. [Wat06, Section 3]).

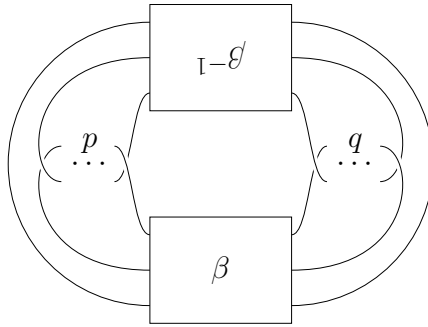


Figure 1: The generalised Kanenobu knot $K_\beta(p, q)$. p and q represent the number of (positive) half twists.

The next theorem is a generalisation of [HW14, Theorem 6.12] to the case of the generalised Kanenobu knots. It summarises the properties that we will use.

Theorem 5. *Let β be a 3-braid such that $K_\beta(p, q)$ is always a knot. Then, for every $p, q \in \mathbb{Z}$ we have*

- (i) $\det(K_\beta(p, q)) = (\det B_\beta)^2$, where B_β is the knot in Figure 2;
- (ii) $\text{Kh}(K_\beta(p, q)) \cong \text{Kh}(K_\beta(p + 1, q - 1))$;
- (iii) $\text{Kh}^{\text{odd}}(K_\beta(p, q)) \cong \text{Kh}^{\text{odd}}(K_\beta(p + 1, q - 1))$;
- (iv) $\text{HFK}(K_\beta(p, q)) \cong \text{HFK}(K_\beta(p + 2, q)) \cong \text{HFK}(K_\beta(p, q + 2))$.

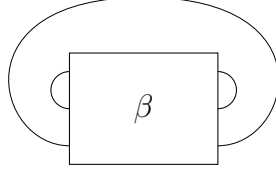


Figure 2: The knot B_β .

By Kh and Kh^{odd} we respectively mean the (bigraded) Khovanov homology and odd Khovanov homology with \mathbb{Z} -coefficients, and by HFK we mean both $\widehat{\text{HFK}}$ and HFK^- (always with the bigrading) with $\mathbb{Z}/2$ -coefficients.

Proof of Theorem 5. We shall postpone the proof of (i). It will be a straightforward consequence of the presentation of the first homology group of the branched double cover that we will derive in Section 2 for the case of $K(n, p, q)$, and that can be generalised to $K_\beta(p, q)$ (cf. Lemmas 8 and 11).

- (ii) It was proved by Watson (cf. [Wat06, Lemma 3.1]).
- (iii) The proof of (iii) is essentially the same as (ii), with a difference in the case of the groups with homological grading 0, where the equality of the Jones polynomial of the two knots (given by (ii)) is used to finish the proof (as explained in [GW11, Theorem 9]).
- (iv) This is an application of [HW14, Theorem 1], where the band is placed in correspondence of the p half twists or the q half twists. \square

Consider now the families of knots

$$\mathcal{F}_\beta(p_0, q_0) = \{K_\beta(p_0 + 2n, q_0 - 2n) \mid n \in \mathbb{Z}\}.$$

By Theorem 5 all the knots in one of these families have the same homological invariants (Khovanov homology, odd-Khovanov homology and knot Floer homology).

A key observation of Greene and Watson in [GW11] says that the d -invariants of the branched double covers of the knots in $\mathcal{F}_\beta(p_0, q_0)$ are related to the coefficients of their Turaev torsion. Specifically, we have the following proposition.

Proposition 6. *Let β be a 3-braid such that $K_\beta(p, q)$ is always a knot, and let p_0 and $q_0 \in \mathbb{Z}$. Then there exists a constant $\lambda \in \mathbb{R}$ such that, for every knot $K \in \mathcal{F}_\beta(p_0, q_0)$ such that the branched double cover $\Sigma(K)$ is an L -space, and for every $\text{Spin}^\mathbb{C}$ structure \mathfrak{t} on $\Sigma(K)$, we have*

$$d(\Sigma(K), \mathfrak{t}) = 2 \cdot \tau(\Sigma(K), \mathfrak{t}, 1_{H_1(\Sigma(K); \mathbb{Z})}) - \lambda.$$

By $\tau(\Sigma(K), \mathfrak{t}, 1_{H_1(\Sigma(K); \mathbb{Z})})$ or $\tau(\Sigma(K), \mathfrak{t}, 1)$ we mean the rational coefficient of 1 of the maximal abelian torsion $\tau(\Sigma(K), \mathfrak{t}) \in \mathbb{Q}[H_1(\Sigma(K); \mathbb{Z})]$ defined in [Tur02, Section I.3]. Notice that we are omitting the homological orientation ω of the 3-manifold $\Sigma(K)$ because every oriented 3-manifold has a canonical homological orientation induced by Poincaré duality (cf. [Tur02, I.4.3]).

Remark 7. A condition that guarantees that all the branched double covers $\Sigma(K)$ (for $K \in \mathcal{F}_\beta(p_0, q_0)$) are L -spaces is that there is one knot $K_\beta(p, q)$ with $p + q = p_0 + q_0$ that is Kh-thin (i.e. the reduced Khovanov homology is torsion-free and supported in a single δ -grading). In this case, by Theorem 5.(ii), all knots K in $\mathcal{F}_\beta(p_0, q_0)$ are Kh-thin, and the spectral sequence from $\text{Kh}(\overline{K})$ to $\widehat{\text{HF}}(\Sigma(K))$ (cf. [OSz05]) implies that $\Sigma(K)$ is an L -space.

Proof of Proposition 6. Since $\Sigma(K)$ is an L -space, we can apply Rustamov's formula (cf. [Rus04, Theorem 3.4] and [GW11, Theorem 12]) to obtain

$$d(\Sigma(K), \mathfrak{t}) = 2 \cdot \tau(\Sigma(K), \mathfrak{t}, 1) - \lambda(\Sigma(K)). \quad (1)$$

Here $\lambda(\Sigma(K))$ is the Casson-Walker invariant, computed by the following formula (cf. [Mul93, Theorem 5.1] and [GW11, Theorem 13]):

$$\lambda(\Sigma(K)) = -\frac{V'_K(-1)}{6 \cdot V_K(-1)} + \frac{\sigma(K)}{4}, \quad (2)$$

where V_K denotes the Jones polynomial and σ denotes the signature.

As all the knots of the form $K_\beta(p, q)$ are ribbon, $\sigma(K) = 0$. Moreover, the Jones polynomial is determined by the Khovanov homology, which is the same for all knots in $\mathcal{F}_\beta(p_0, q_0)$ (cf. Theorem 5.(ii)). Thus, Equation (2) shows that $\lambda(\Sigma(K))$ is a constant λ , so Equation (1) concludes the proof. \square

The goal of this paper is to prove that for every integer $n \geq 2$ there are infinite families of knots with determinant $(2n+1)^2$ and the same homological invariants, such that the d -invariants of their branched double covers are not bounded from above or below. To achieve it, we will compute the Turaev torsion of the branched double covers for some families $\mathcal{F}_\beta(p_0, q_0)$ and we will see that they are unbounded. Then we will apply Proposition 6, that implies that the d -invariants are unbounded if and only if the coefficients of the Turaev torsion are as well.

To simplify our computations, we will focus on a particular family of braids, namely the braids $\beta_n = \sigma_1 \sigma_2^{-1} \sigma_1^n$, with $n \geq 2$, represented in Figure 3.

For the rest of the paper we will write $K(n, p, q)$ for $K_{\beta_n}(p, q)$ and $\mathcal{F}(n, p_0, q_0)$ for $\mathcal{F}_{\beta_n}(p_0, q_0)$. Also, $\Sigma(n, p, q)$ will denote the branched double cover $\Sigma(K_{\beta_n}(p, q))$.

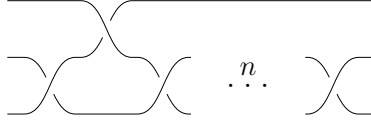


Figure 3: The 3-braid $\beta_n = \sigma_1 \sigma_2^{-1} \sigma_1^n$, with $n \geq 2$.

2. A PRESENTATION OF $\pi_1(\Sigma(n, p, q))$ AND $H_1(\Sigma(n, p, q))$

In this section we find a presentation of the group $\pi_1(\Sigma(n, p, q))$ and a presentation of the \mathbb{Z} -module $H_1(\Sigma(n, p, q))$, for every $K(n, p, q)$.

2.1. A presentation of $\pi_1(\Sigma(n, p, q))$

The method we use to find a presentation of the fundamental group of $\Sigma(n, p, q)$ relies on the algorithm explained in [Gre13, Section 3], that we briefly recall. Start from a planar diagram of a knot K and colour the complement of the projection of the knot in a chessboard fashion, in such a way that the unbounded region is white. Construct the white graph as follows: for every white region draw a vertex, and for every crossing draw an edge between the two adjacent white regions. If you now remove the vertex associated to the unbounded region (but not the edges emanating from it), what is left is called the *reduced white graph* of the projection. Label the vertices of the reduced white graph by e_1, \dots, e_w , and label each edge with the sign of the associated crossing (according to the convention as in Figure 4). Now fix a vertex e_i ; for every edge emanating from e_i to e_j record a word $(e_j e_i^{-1})^\varepsilon$, and for every edge emanating from e_i to the unbounded region record the word $e_i^{-\varepsilon}$, where ε is the sign of the edge. Let b_i be the word obtained by concatenating the words associated to all edges emanating from e_i , recorded by counting counterclockwise. Then, a presentation of $\pi_1(\Sigma(K))$ is

$$\pi_1(\Sigma(K)) = \langle e_1, \dots, e_w \mid b_1, \dots, b_w \rangle.$$

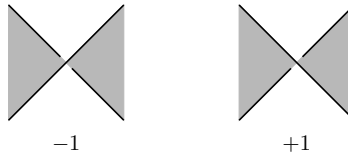


Figure 4: The sign associated to a crossing.

In the case of the knot $K(n, p, q)$, the reduced white graph looks as in Figure 5. Therefore, we have that

$$\pi_1(\Sigma(n, p, q)) = \langle e_1, e_2, e_3, e_4 \mid b_1, b_2, b_3, b_4 \rangle, \quad (3)$$

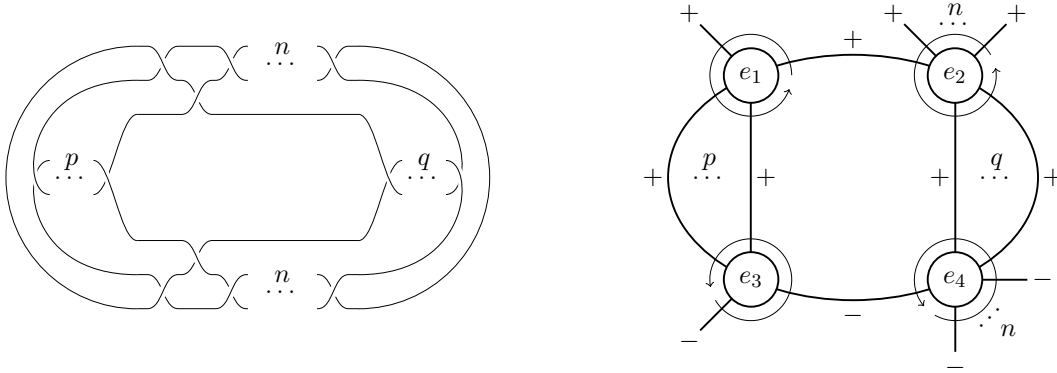


Figure 5: A diagram of the knot $K(n, p, q)$ (on the left) and its associated reduced white graph (on the right).

where

$$b_1 = e_2 e_1^{-2} (e_3 e_1^{-1})^p; \quad (4a)$$

$$b_2 = e_2^{-n} e_1 e_2^{-1} (e_4 e_2^{-1})^q; \quad (4b)$$

$$b_3 = e_3^2 e_4^{-1} (e_1 e_3^{-1})^p; \quad (4c)$$

$$b_4 = e_4^n (e_2 e_4^{-1})^q e_4 e_3^{-1}. \quad (4d)$$

2.2. A presentation of $H_1(\Sigma(n, p, q))$

In order to obtain a presentation of $H_1(\Sigma(n, p, q))$ it is sufficient to abelianise the presentation of Equation (3). Thus, a presentation matrix for $H_1(\Sigma(n, p, q))$ is

$$M_{n,p,q} = \begin{pmatrix} -p-2 & 1 & p & 0 \\ 1 & -q-n-1 & 0 & q \\ p & 0 & -p+2 & -1 \\ 0 & q & -1 & -q+n+1 \end{pmatrix}.$$

Now we can prove Theorem 2.(i) in the case of the knots $K(n, p, q)$:

Lemma 8. *For every $n \geq 2$, $p, q \in \mathbb{Z}$, we have*

$$\det(K(n, p, q)) = (2n+1)^2;$$

$$\det(B_{\beta_n}) = 2n+1.$$

Proof. For the first equality, $\det(K(n, p, q)) = |\det(M_{n,p,q})| = (2n+1)^2$.

In the same way as we found a presentation matrix for $H_1(\Sigma(n, p, q))$, we can derive one for $H_1(\Sigma(B_{\beta_n}))$ by using the diagram in Figure 2. We then find that a presentation matrix for $H_1(\Sigma(B_{\beta_n}))$ is given by the bottom-right 2×2 minor of the matrix $M_{n,0,0}$. The determinant of this minor is $2n+1$. \square

The computation of the Turaev torsion will be much easier when $H_1(\Sigma(n, p, q))$ is cyclic. Therefore, we will now state in the following lemma a condition that guarantees that this group is cyclic.

Lemma 9. $H_1(\Sigma(n, p, q))$ is cyclic if and only if $\gcd(2q + (n + 1)p, 2n + 1) = 1$. Moreover, if it is cyclic, both $[e_2]$ and $[e_4]$ are generators.

Proof. After some column moves, the matrix $M_{n,p,q}$ has the form

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -2q - (n + 1)p - (2n + 1) & -q - (n + 1) & -(2n + 1) & q \\ 0 & 0 & 0 & -1 \\ 2q + (n + 1)p & q & 2n + 1 & -q + (n + 1) \end{pmatrix}. \quad (5)$$

Thus, another presentation matrix for $H_1(\Sigma(n, p, q))$ is obtained by taking the minor

$$\begin{pmatrix} -2q - (n + 1)p - (2n + 1) & -(2n + 1) \\ 2q + (n + 1)p & 2n + 1 \end{pmatrix}. \quad (6)$$

By a standard argument of Commutative Algebra, it is clear from the above presentation that $H_1(\Sigma(n, p, q))$ is cyclic if and only if $\gcd(2q + (n + 1)p, 2n + 1) = 1$. In such a case, as the generators in the presentation of Equation (6) are $[e_2]$ and $[e_4]$, each of them generates the whole abelian group. \square

Remark. The condition $\gcd(2q + (n + 1)p, 2n + 1) = 1$ of Lemma 9 is equivalent to $\gcd(4q + p, 2n + 1) = 1$ (by multiplying $2q + (n + 1)p$ by 2).

Recall that we are interested in studying the knots in the family $\mathcal{F}(n, p_0, q_0)$, and we would like to rule out the knots such that their branched double covers do not have cyclic homology. If we define the family

$$\tilde{\mathcal{F}}(n, p_0, q_0) = \{K(n, p, q) \in \mathcal{F}(n, p_0, q_0) \mid \gcd(p + 4q, 2n + 1) = 1\}, \quad (7)$$

then by Lemma 9 the first homology group of the branched double cover of any knot in $\tilde{\mathcal{F}}(n, p_0, q_0)$ is cyclic.

What we have to check is that $\tilde{\mathcal{F}}(n, p_0, q_0)$ is still an infinite family. Since $p + q = p_0 + q_0$, the condition $\gcd(p + 4q, 2n + 1) = 1$ becomes

$$\gcd(3q + p_0 + q_0, 2n + 1) = 1. \quad (8)$$

We now separate into two cases.

If $3 \nmid 2n + 1$, then there are $\phi(2n + 1)$ solutions modulo $(2n + 1)$ to Equation (8), so the family $\tilde{\mathcal{F}}(n, p_0, q_0)$ is still infinite in both directions (i.e. with $p \gg 0$ and $p \ll 0$).

If instead $3 \mid 2n + 1$, then either $p_0 + q_0$ is divisible by 3, in which case Equation (8) is impossible, or $\gcd(p_0 + q_0, 3) = 1$. In the latter case, write $2n + 1 = 3^a \cdot b$, with $3 \nmid b$. Then Equation (8) is equivalent to

$$\gcd(3q + p_0 + q_0, b) = 1.$$

The number of solutions modulo $(2n + 1)$ in this case is

$$3^a \cdot \phi(b) \geq 3,$$

where the Euler function ϕ is redefined at 1 in such a way that $\phi(1) = 1$.

We can summarise our computation in the following remark.

Remark 10. If $3 \nmid \gcd(2n + 1, p_0 + q_0)$, then the family $\tilde{\mathcal{F}}(n, p_0, q_0)$ consists of infinitely many knots (with p arbitrarily large and arbitrarily small). Moreover,

- (i) for every $K \in \tilde{\mathcal{F}}(n, p_0, q_0)$, $\det K = (2n + 1)^2$;
- (ii) all knots in $\tilde{\mathcal{F}}(n, p_0, q_0)$ have isomorphic Khovanov, odd-Khovanov and knot Floer homology;
- (iii) for every $K \in \tilde{\mathcal{F}}(n, p_0, q_0)$, $H_1(\Sigma(K)) \cong \mathbb{Z}/((2n + 1)^2)$, generated by $[e_4]$.

Before concluding this section, for the sake of completeness we remark that the same computations of this section for a general braid β yield the following generalisation of Lemmas 8 and 9, that concludes the proof of Theorem 5.(i).

Lemma 11. *For all 3-braids β such that $K_\beta(p, q)$ is always a knot, and for all $p, q \in \mathbb{Z}$, we have $\det(K_\beta(p, q)) = (\det B_\beta)^2$.*

Also, $H_1(\Sigma(K_\beta(p, q))) \cong \mathbb{Z}/(\det B_\beta)^2$ if and only if $\gcd(q\Delta_m + p\Delta_1, \det B_\beta) = 1$, where Δ_1 and Δ_m are suitable minors of the presentation matrix for $H_1(\Sigma(B_\beta))$ arising from the reduced white graph of the diagram in Figure 2.

3. COMPUTING THE TURAEV TORSION

In this section we will prove the following lemma.

Lemma 12. *Suppose that $3 \nmid \gcd(2n + 1, p_0 + q_0)$. Then for every $K(n, p, q)$ in $\tilde{\mathcal{F}}(n, p_0, q_0)$ there exist $\text{Spin}^\mathbb{C}$ structures \mathfrak{s}_p and \mathfrak{s}'_p on $\Sigma(n, p, q)$ such that for $p \rightarrow \pm\infty$ we have*

$$\begin{aligned} \tau(\Sigma(n, p, q), \mathfrak{s}_p, 1) &\rightarrow \pm\infty; \\ \tau(\Sigma(n, p, q), \mathfrak{s}'_p, 1) &\rightarrow \mp\infty. \end{aligned}$$

Before proving the lemma, let us recall the properties of the Turaev torsion that we will need. We follow [Tur02, Section I.3], focusing on the case of rational homology spheres (or - equivalently - closed connected oriented 3-manifolds with finite first homology group). The maximal abelian (Turaev) torsion of a rational homology sphere Y (with the standard homological orientation) and a $\text{Spin}^\mathbb{C}$ structure $\mathfrak{t} \in \text{Spin}^\mathbb{C}(Y)$ is an element of $\mathbb{Q}[H]$, where $H = H_1(Y; \mathbb{Z})$. We can write it in terms of its rational coefficients as

$$\tau(Y, \mathfrak{t}) = \sum_{h \in H} \tau(Y, \mathfrak{t}, h) \cdot h.$$

Recall that ([Tur02, Chapter I, Equation (3.c)])

$$\tau(Y, g \cdot \mathfrak{t}, h) = \tau(Y, \mathfrak{t}, g^{-1}h),$$

where the action of H on $\text{Spin}^{\mathbb{C}}(Y)$ is the usual free and transitive action of $H_1(Y; \mathbb{Z})$ on $\text{Spin}^{\mathbb{C}}(Y)$. Thus, it is sufficient to know $\tau(Y, \mathbf{t}, 1)$ for every \mathbf{t} to determine the maximal abelian torsion.

Moreover, recall that any surjective ring homomorphism $\varphi : \mathbb{Q}[H] \rightarrow \mathbb{K}$, where \mathbb{K} is a cyclotomic field, carries the maximal abelian torsion to the φ -twisted torsion:

$$\varphi(\tau(Y, \mathbf{t})) = \tau^{\varphi}(Y, \mathbf{t}).$$

Thus, the rational coefficients of the φ -twisted torsion are sums of the rational coefficients of the maximal abelian torsion. Also, the sum of all the rational coefficients $\tau(Y, \mathbf{t}, 1)$ vanishes because it is the twisted torsion associated to the augmentation map $\mathbb{Q}[H] \rightarrow \mathbb{Q}$ (cf. [Tur02, I.1.2]).

In [Tur02, Theorem II.1.2, case (4)] Turaev describes a method to compute the φ -twisted torsion which we summarise in the following remark.

Remark 13. Let Y be a 3-dimensional closed connected oriented manifold, and let E be a cellular decomposition of Y with one 0-cell, w 1-cells e_1, \dots, e_w , w 2-cells f_1, \dots, f_w , and one 3-cell. Suppose that every cell is endowed with an orientation. Choose relations b_i that represent the boundaries of the 2-cells in their homotopy classes.

Let $A = ([\partial_i b_j])_{i,j}$ be the matrix of the abelianised Fox derivatives of the relations b_j , and let $\Delta^{r,s}$ denote the determinant of the minor of A obtained by deleting the s -th row and the r -th column.

Let h_s be the homology class of the 1-cell e_s in $H = H_1(Y; \mathbb{Z})$, and let $g_r \in H$ be the homology class of a loop in Y that intersects once positively the 2-cell f_r and is disjoint from the other 2-cells.

Then, there exists an Euler structure \mathbf{t} on Y such that, for each ring homomorphism $\varphi : \mathbb{Q}[H] \rightarrow \mathbb{K}$ with $\varphi(g_r - 1) \neq 0$, $\varphi(h_s - 1) \neq 0$ and $\varphi(\Delta^{r,s}) \neq 0$, the φ -torsion is given by

$$\tau^{\varphi}(Y, \mathbf{t}) = \pm \frac{\varphi(\Delta^{r,s})}{\varphi(h_s - 1) \varphi(g_r - 1)}. \quad (9)$$

Proof of Lemma 12. We apply Remark 13 to the case of Y being $\Sigma(n, p, q)$, the double branched cover of the knot $K(n, p, q) \in \tilde{\mathcal{F}}(n, p_0, q_0)$. The loops e_i and the relations b_i are the ones that appear in Equation (3). We choose $r = s = 4$. Then $h_4 = e_4$ and $g_4 = e_4^{-1}$.

By Remark 10.(iii) we know that $H \cong \mathbb{Z}/((2n+1)^2)$ and that $[e_4]$ is a generator of H . Thus, we can define the ring homomorphism

$$\varphi : \mathbb{Q}[H] \cong \mathbb{Q} \left[\mathbb{Z}/((2n+1)^2) \right] \longrightarrow \mathbb{Q}(\zeta) = \mathbb{K}$$

that carries $[e_4]$ to some $(2n+1)$ -th primitive root of unity ζ .

Then, Remark 13 says that (provided that $\varphi(\Delta^{4,4}) \neq 0$) there exists a $\text{Spin}^{\mathbb{C}}$ structure \mathbf{t}_p such that

$$\tau^{\varphi}(\Sigma(n, p, q), \mathbf{t}_p) = \frac{\pm 1}{(\zeta - 1) \cdot (\zeta^{-1} - 1)} \cdot \varphi(\Delta^{4,4}) = R \cdot \varphi(\Delta^{4,4}), \quad (10)$$

where R is a real number $\neq 0$.

In order to compute $\varphi(\Delta^{4,4})$, we have to understand what the image of every $[e_i]$ is. Recall that after some column moves, the matrix $M_{n,p,q}$ appears as in Equation (5).

The relation given by the third column implies that

$$\varphi([e_2]) = \varphi([e_4]) = \zeta$$

and the relations given by the second and the fourth columns imply that

$$\varphi([e_1]) = \varphi([e_3]) = \zeta^{n+1}.$$

A computation then shows that, if A is the matrix of the abelianised Fox derivatives, $\varphi(A)$ is of the form

$$\begin{pmatrix} -p-1-\zeta^{n+1} & \zeta^{n+1} & p & \varphi([\partial_1 b_4]) \\ 1 & -q - \frac{1-\zeta^{-(n+1)}}{1-\zeta^{-1}} & 0 & \varphi([\partial_2 b_4]) \\ p & 0 & 1+\zeta^{n+1}-p & \varphi([\partial_3 b_4]) \\ \varphi([\partial_4 b_1]) & \varphi([\partial_4 b_2]) & \varphi([\partial_4 b_3]) & \varphi([\partial_4 b_4]) \end{pmatrix}.$$

From this, one can easily compute $\varphi(\Delta^{4,4})$ in terms of p and q . By using the relation $q = -p + p_0 + q_0$, we obtain that there is some constant $C_1 \in \mathbb{Q}(\zeta)$ such that

$$\varphi(\Delta^{4,4}) = -(\zeta^{n+1} + \zeta + 1)p + C_1.$$

If $n \geq 2$ we have that $\zeta^{n+1} + \zeta + 1 \neq 0$, so $\varphi(\Delta^{4,4})$ vanishes for at most one value of p . For all other p , by Equation (10) we obtain that

$$\tau^\varphi(\Sigma(n, p, q), \mathfrak{t}_p) = -R(\zeta^{n+1} + \zeta + 1)p + C_2,$$

for some constant $C_2 \in \mathbb{Q}(\zeta)$. Notice that the coefficient of p is non-zero if $n \geq 2$.

We now see that the torsion $\tau^\varphi(\Sigma(n, p, q), \mathfrak{t}_p)$ is the sum of a constant term C_2 and of a term that varies linearly in p . Together with the fact that the sum of all the rational coefficients of $\tau^\varphi(\Sigma(n, p, q), \mathfrak{t}_p)$ is 0 (cf. [Tur02, I.1.2]), we deduce that there must exist $\text{Spin}^{\mathbb{C}}$ structures \mathfrak{s}_p and \mathfrak{s}'_p such that for $p \rightarrow \pm\infty$ we have

$$\tau(\Sigma(n, p, q), \mathfrak{s}_p, 1) \rightarrow \pm\infty;$$

$$\tau(\Sigma(n, p, q), \mathfrak{s}'_p, 1) \rightarrow \mp\infty. \quad \square$$

4. THE FINAL STEP

In order to show that there are families of knots with unbounded d -invariants, we would like to apply Proposition 6. However, we need all the branched double covers to be L -spaces. By Remark 7 it is sufficient to look for thin knots. Recall that a knot is (homologically) thin if its reduced Khovanov and odd-Khovanov homologies with \mathbb{Z} -coefficients and its knot Floer homology with $\mathbb{Z}/2$ -coefficients are free modules and are supported in a single δ -grading. Some thin knots $K(n, p, q)$ can be found by applying [CO12, Theorem 5.3], as explained in the following lemma.

Lemma 14. *For every $n \geq 2$ and for every q_0 such that $|q_0| < n + 1$, $K(n, 0, q_0)$ is quasi-alternating, hence thin.*

For a definition of quasi-alternating links, see [OSz05, Definition 3.1].

Proof. For the case of $q_0 = 0$, we prove that $K_\beta(0, 0)$ is alternating for every β . Notice that $K_\beta(0, 0) = B_\beta \# \overline{B_\beta}$. B_β and $\overline{B_\beta}$ are alternating because they are 2-bridge knots (cf. [Goo72]), so $K_\beta(0, 0)$ is alternating as well. As every alternating knot is quasi-alternating (cf. [OSz05, Lemma 3.2]), and every quasi-alternating knot is thin (cf. [MO08]), this concludes the proof in the case of $q_0 = 0$.

When $q_0 \neq 0$, the knot $K(n, 0, q_0)$ is a Montesinos knot. [CO12, Theorem 5.3] then implies that $K(n, 0, q_0)$ is quasi-alternating (hence thin) when $0 \neq |q_0| < n + 1$. \square

We can now prove Theorem 1, that we restate here in a more precise form.

Theorem 1. *Let $n \geq 2$ and q_0 be integers satisfying $3 \nmid \gcd(q_0, 2n + 1)$ and $|q_0| < n + 1$. Then the family $\tilde{\mathcal{F}}(n, 0, q_0)$ is an infinite family of thin knots with determinant $(2n + 1)^2$, same Khovanov, odd-Khovanov and knot Floer homologies, and for every $K(n, p, q) \in \tilde{\mathcal{F}}(n, 0, q_0)$ there exist $\text{Spin}^{\mathbb{C}}$ structures \mathfrak{s}_p and \mathfrak{s}'_p on $\Sigma(n, p, q)$ such that for $p \rightarrow \pm\infty$ we have*

$$d(\Sigma(n, p, q), \mathfrak{s}_p) \rightarrow \pm\infty; \quad (11a)$$

$$d(\Sigma(n, p, q), \mathfrak{s}'_p) \rightarrow \mp\infty. \quad (11b)$$

Proof. By Remark 10, the family $\tilde{\mathcal{F}}(n, 0, q_0)$ consists of infinitely many knots with determinant $(2n + 1)^2$ and the same homological invariants. Lemma 14 and Theorem 5 prove that they are all thin. Therefore, their branched double covers are L -spaces (see Remark 7). Thus, we can apply Proposition 6, which, together with Lemma 12, proves the result. \square

A first consequence of Theorem 1 is Theorem 2, that we restate here.

Theorem 2. *For every odd number $\Delta \geq 5$, there exist infinite families of non-quasi-alternating thin knots with determinant Δ^2 and the same homological invariants.*

Proof. Consider the families given by Theorem 1, with $n = \frac{\Delta-1}{2}$. By [GW11, Proposition 3] only finitely many knots in each family can be quasi-alternating. \square

Remark. We can also require the knots in Theorem 2 to be hyperbolic. This can be achieved by using a result by Riley (cf. [Ril79, Corollary, page 102]), in a similar way as in [Kan86, Lemma 5] and [GW11, Proposition 11]. First notice that the bridge number of $K(n, p, q)$ is ≤ 3 (it is clear from the diagram in Figure 5). As explained in the proof of Theorem 2, for $p \gg 0$ and $p \ll 0$ the knots $K(n, p, q)$ in $\tilde{\mathcal{F}}(n, 0, q_0)$ are not quasi-alternating. In particular, they are not alternating and hence they are not 2-bridge. Thus, for each $n \geq 2$ and for each q_0 such that $|q_0| < n + 1$ the bridge number of all but finite knots in $\tilde{\mathcal{F}}(n, 0, q_0)$ must be 3. Riley's theorem (cf. [Ril79,

Corollary, page 102]) then implies that such knots are either composite, torus knots or hyperbolic. The possibility of being composite is ruled out by the fact that the first homology group is cyclic, as in [GW11, Proposition 11]. Moreover, $K(n, p, q)$ is never a torus knot because it is slice. Thus, for each $n \geq 2$ and for each q_0 such that $|q_0| < n + 1$ all but finite knots in $\tilde{\mathcal{F}}(n, 0, q_0)$ are hyperbolic.

5. MANIFOLDS THAT ARE NOT SURGERY ON A KNOT IN S^3

The aim of this last section is to prove Theorem 3, which we restate below, as an application of Theorem 1. The techniques come from [Doi12] and [HW13].

Theorem 3. *For every odd integer $\Delta \gg 0$, there exist infinitely many hyperbolic, weight 1 manifolds $M_{\Delta, p}$ with $|H_1(M_{\Delta, p})| = \Delta^2$ that are not surgery on a knot in S^3 .*

Recall that a manifold is called weight 1 if its fundamental group is the normal closure of one element. We split the proof of the theorem in 3 lemmas.

Lemma 15. *For every $n \gg 0, |p| \gg 0, |q| \gg 0$, the manifold $\Sigma(n, p, q)$ is hyperbolic.*

Proof. The manifold $\Sigma(n, p, q)$ is obtained by quadruple Dehn filling on the double branched cover $\Sigma(T)$ of the tangle T in Figure 6. If $\Sigma(T)$ is hyperbolic, then by [Thu79, Theorem 5.8.2] so are the manifolds $\Sigma(n, p, q)$ for $n, |p|$ and $|q|$ big enough.

To check the hyperbolicity of $\Sigma(T)$, we input the tangle T into the computer software Orb (cf. [Hea05]). As explained in [HW13, Proof of Lemma 4.7], Orb can find a triangulation of $\Sigma(T)$, which can be easily detected, among the options that Orb gives, by its first homology¹. Such triangulation consists of 10 tetrahedra and has degree 6 at every edge. Thus, the choice of $e^{\frac{\pi i}{3}}$ as shape parameter for each tetrahedron gives a hyperbolic structure to $\Sigma(T)$. This hyperbolic structure is complete if the cusp equations, which are of the form

$$z_1^{a_1} \cdot \left(\frac{1}{1-z_1}\right)^{b_1} \cdot \left(\frac{z_1-1}{z_1}\right)^{c_1} \cdot \dots \cdot z_{10}^{a_{10}} \cdot \left(\frac{1}{1-z_{10}}\right)^{b_{10}} \cdot \left(\frac{z_{10}-1}{z_{10}}\right)^{c_{10}} = 1$$

and are thought of as equations on the universal cover of \mathbb{C}^* , are satisfied by the choice of shape parameters given by $z_j = e^{\frac{\pi i}{3}}$ for every $j = 1, \dots, 10$. In our case there are 8 cusp equations (computed with SnapPy [CDW]), represented by the rows of the following matrix, that we interpret as vectors $(a_1, b_1, c_1, \dots, a_{10}, b_{10}, c_{10})$:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & -2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}.$$

¹ $H_1(\Sigma(T); \mathbb{Z})$ can be computed as follows. The right part of Figure 6 shows a quadruple tangle filling turning T into the unknot. The Montesinos trick then implies that $\Sigma(T)$ is obtained by S^3 by removing 4 solid tori, i.e. it is a 4-component link complement (specifically, SnapPy [CDW] identifies $\Sigma(T)$ with the complement of the link L10n101). By Alexander duality we have $H_1(\Sigma(T); \mathbb{Z}) \cong \mathbb{Z}^{\oplus 4}$.

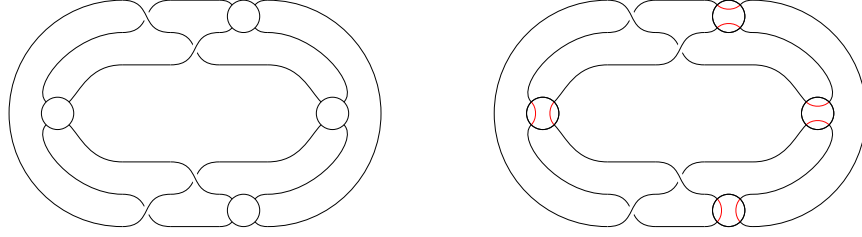


Figure 6: The tangle T (on the left) and a filling yielding the unknot (on the right).

It is straightforward to check that $z_j = e^{\frac{\pi i}{3}}$ is a solution for all cusp equations. Thus, the manifold $\Sigma(T)$ admits a complete hyperbolic structure. \square

Lemma 16. *For all integers n, p, q such that $\gcd(2q + (n + 1)p, 2n + 1) = 1$, the manifold $\Sigma(n, p, q)$ is weight 1.*

Proof. We prove that $G = \pi_1(\Sigma(n, p, q)) / \langle\langle e_4 \rangle\rangle$ is trivial. A presentation for G is obtained from (3) by adding the relation $e_4 = 1$ to the set of relations (4). First, the relations b_2 and b_4 respectively become $e_1 = e_2^{q+n+1}$ and $e_3 = e_2^q$. Then, the relations b_1 and b_3 respectively become $e_2^{2q+(n+1)p+(2n+1)} = 1$ and $e_2^{2q+(n+1)p} = 1$. As $\gcd(2q + (n + 1)p, 2n + 1) = 1$, we obtain $e_2 = 1$, from which we deduce that the group G is trivial. \square

Lemma 17. *Let $n \geq 2$ and q_0 be integers satisfying $3 \nmid \gcd(q_0, 2n + 1)$ and $|q_0| < n + 1$. Then, only for finitely many knots $K(n, p, q)$ in the family $\tilde{\mathcal{F}}(n, 0, q_0)$ the branched double cover $\Sigma(n, p, q)$ is surgery on a knot in S^3 .*

Proof. Let $K(n, p, q) \in \tilde{\mathcal{F}}(n, 0, q_0)$, and suppose that $\Sigma(n, p, q)$ is a surgery on some knot $K \in S^3$ with slope $\frac{r}{s} \geq 0$ (we will deal with the case of negative slope afterwards). First notice that, by homology, $r = (2n + 1)^2$.

By [NW10, Theorem 2.5], there is an identification $\text{Spin}^{\mathbb{C}}(S_{r/s}^3(K)) \cong \mathbb{Z}/r \cong \text{Spin}^{\mathbb{C}}(L(r, s))$ such that for all $i \in \mathbb{Z}/r$

$$d(S_{r/s}^3(K), i) \leq d(L(r, s), i). \quad (12)$$

As there are only finitely many lens spaces with first homology \mathbb{Z}/r , $d(L(r, s), i)$ can only take finitely many values. This gives an upper bound to each d -invariant $d(S_{r/s}^3(K), i)$. In view of Equations (11), only for finitely many $K(n, p, q) \in \tilde{\mathcal{F}}(n, 0, q_0)$, the manifold $\Sigma(n, p, q)$ can be $S_{r/s}^3(K)$ for some knot K in S^3 and some $\frac{r}{s} \geq 0$.

For the case of negative surgery, it is sufficient to notice that if $\Sigma(n, p, q)$ is a negative surgery on a knot K , then $-\Sigma(n, p, q)$ is a positive surgery on \overline{K} , the mirror image of K . Now the equality $d(-Y, t) = -d(Y, t)$ (cf. [OSz03, Proposition 4.2]) and Equation (12) give a lower bound to the d -invariants of $\Sigma(n, p, q)$. By Equations (11) this is possible only for finitely many $K(n, p, q)$. \square

Proof of Theorem 3. Consider a family $\tilde{\mathcal{F}}(n, 0, q_0)$ satisfying the same hypotheses of Theorem 1. Define $M_{\Delta, p} = \Sigma(n, p, q_0 - p)$, where $\Delta = 2n + 1$. Lemmas 15, 16 and 17 imply that, for $|p| \gg 0$ and $n \gg 0$, $M_{\Delta, p}$ is hyperbolic, weight 1 and it is not surgery on a knot in S^3 . \square

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